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States on Partial Rings

M. Lynn Krause[1] and Gottfried T. Rüttimann[1]

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Recently, the structure theory of JB*-triples has received considerable attention. The reason is that JB*-triples and those JB*-triples which are dual spaces, the JBW*-triples, not only form natural generalizations of Jordan C*-algebras and C*-algebras, and Jordan W*-algebras and W*-algebras, but also provide a context for the study of infinite-dimensional holomorphy and infinite-dimensional Lie algebras. In a JBW*-triple the tripotents play the role of the projections in a W*-algebra. In analogy to the projection lattice of a W*-algebra, we investigate the partial ring of tripotents of JBW*-triple. Unlike on W*-algebras, states, i.e., positive normalized homomorphisms from the partial ring of tripotents of a JBW*-triple into the partial ring of real numbers, have not yet been discussed in the literature. We show that the partial ring of tripotents of a JBW*-triple admits a unital set of Jauch–Piron states.

1. INTRODUCTION

Of late, in the context of quantum mechanical probability theory, the algebraic structures studied have been of steadily increasing generality. Effect algebras, which can contain isotropic elements, generalize orthoalgebras. S-sets, in which no cancellation law is stipulated, generalize effect algebras etc. (see, e.g., (Foulis and Bennett, 1994; Foulis *et al.*, 1992, 1993; Gudder, 1995; Hedlíkova and Pulmannová, n.d.). In this paper, we approach the problem so to speak from the opposite end, beginning with what is, in a certain sense, the most general structure possible, and examining what results when this structure becomes more specific in various ways.

The key concept of our approach is that of a partial ring, which serves as the basic event structure on which to build a noncommutative nonassociative probability theory. Section 2 is devoted to the algebraic part of partial rings. In section 3 we develop the functional analytic framework of measure spaces on partial rings in the tradition of Navara and Rüttiman (1991), Rüttimann

[1] University of Berne, Berne, Switzerland.

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(1989, 1994), Rüttiman and Schindler (1987), and Rüttimann and Wright (1995).

In the last few years, Jordan triple systems have gained considerable attention as algebraic structures which embrace both noncmmuative and nonassociative algebras. JB*-triples and JBW*-triples not only accommodate C*-algebras and W*algebras, JB*-algebras and JBW*-algebras, etc., they also establish a link with bounded symmetric domains.

In Section 4 we investigate the partial ring of tripotents of a JBW*-triple and prove the existence of a unital set of complete Jauch–Piron states on this partial ring.

2. PARTIAL RINGS

A triple $(U, \perp, \oplus)$, where U is a set, $\perp$ is a subset of $U \times U$, and $\oplus$ is a map from $\perp$ into U, is said to a *partial ring.*

Let $(U, \perp, \oplus)$ be a partial ring. If the pair (u, v) of elements in U lies in $\perp$, then we write $u \perp v$ and call the pair (u, v) *orthogonal.* For an element (u, v) in $U \times U$ we write $u \oplus v$ to assert that $u \perp v$ and to denote the image of (u, v) under the map $\oplus$. For an orthogonal pair (u, v) in U, the element $u \oplus v$ is called the *sum of u and v.*

A partial ring U is said to satisfy the *right cancellation law* if, for all elements u, v, w in U,

$$u \perp v, \qquad u \perp w, \qquad \text{and} \qquad u \oplus v = u \oplus w \Rightarrow v = w$$

The partial ring U is said to satisfy the *left cancellation law* if for all elements u, v, w in U,

$$u \perp v, \qquad w \perp v, \qquad \text{and} \qquad u \oplus v = w \oplus v \Rightarrow u = w$$

A partial ring which satisfies the left and the right cancellation laws is said to satisfy the *cancellation law.*

A partial ring U is said to satisfy the *right associative law* if, for all elements u, v, w in U, $u \perp v$ and $(u \oplus v) \perp w$ implies $v \perp w$, $u \perp (v \oplus w)$ and $(u \oplus v) \oplus w = u \oplus (v \oplus w)$. The partial ring U is said to satisfy the *left associative law* if, for all elements u, v, w in U, $v \perp w$ and $u \perp (v \oplus w)$ implies $u \perp v$, $(u \oplus v) \perp w$ and $u \oplus (v \oplus w) = (u \oplus v) \oplus w$. The partial ring U is said to satisfy the *associative law* if it satisfies both the right and left associative laws.

A partial ring $(U, \perp, \oplus)$ is called *symmetric or commutative* if, for all elements $u, v \in U$ with $u \perp v$,

$$v \perp u \qquad \text{and} \qquad u \oplus v = v \oplus u$$

holds true. A symmetric partial ring satisfies the right cancellation law if an only if it satisfies the left cancellation law if and only if it satisfies the cancellation law. Likewise, it satisfies the right associative law if an only if it satisfies the left associative law if and only if it satisfies the associative law.

Let $(U, \perp, \oplus)$ be a partial ring. An element 0_r, resp. 0_l, is said to be *right neutral*, resp. *left neutral*, if for all $u \in U$, $u \perp 0_r$, resp. $0_l \perp u$, and $u \oplus 0_r = u$, resp. $0_l \oplus u = u$. Clearly, in a partial ring which admits a right neutral element 0_r and a left neutral element 0_l we have $0_r = 0_l$. An element 0 which is both right neutral and left neutral is called *neutral.* A right neutral element 0_r, resp. a left neutral element 0_l, is called *indecomposable* if, for elements u, v in U with $u \perp v$,

$$0_r = u \oplus v \Rightarrow v = 0_r, \qquad \text{resp.} \quad 0_l = u \oplus v \Rightarrow u = 0_l$$

An element 1_r in U is said to be a *right unit*, resp. *left unit*, in U if, for all elements u in U, there exists an element v in U such that $u \perp v$, resp. $v \perp u$, and $u \oplus v = 1_r$, resp. $v \oplus u = 1_l$. An element 1 which is both a right unit and a left unit is called a *unit.*

Let $(U, \perp, \oplus)$ and $(V, \perp, \oplus)$ be partial rings. A map $\phi\colon U \to V$ is said to be a *homomorphism from U to V* provided that, for all elements u, v in U with $u \perp v$, $\phi(u) \perp \phi(v)$ and $\phi(u \oplus v) = \phi(u) \oplus \phi(v)$. A homomorphism $\phi\colon U \to V$ from U to V is said to be *strict* if, for elements u, v in U, $\phi(u) \perp \phi(v)$ implies that $u \perp v$. Clearly, the identity map of U is a strict homomorphism.

Let $(U, \perp, \oplus)$ be a partial ring. We define binary relations $\leq_r$, $\leq_l$, and $\leq$ on U, for elements u and v in U, by

$$u \leq_r v :\Leftrightarrow \exists w \in U \quad \text{such that } u \perp w \text{ and } u \oplus w = v$$

$$u \leq_l v :\Leftrightarrow \exists w \in U \quad \text{such that } w \perp u \text{ and } w \oplus u = v$$

$$u \leq v :\Leftrightarrow u \leq_r v \quad \text{and} \quad u \leq_l v$$

If the partial ring U is symmetric, then the three binary relations coalesce.

Theorem 2.1. Let $(U, \perp, \oplus)$ be a partial ring which contains an indecomposable right neutral element 0_r, resp. left neutral element 0_l, resp. neutral element 0. If U satisfies the right cancellation law, resp. left cancellation law, resp. cancellation law, and the right associative law, resp. left associative law, resp. associative law, then the binary relation $\leq_r$, resp. $\leq_l$, resp. $\leq$, defined above is a partial ordering on U. A right unit 1_r, resp. left unit 1_l, resp. unit 1, in U is the greatest element in the partially ordered set $(U, \leq_r)$, resp. $(U, \leq_l)$, resp. $(U, \leq)$. Moreover, if U satisfies both the associative law and the cancellation law, then the neutral element 0 is the smallest element in the partially ordered set $(U, \leq)$.

Proof. This is straightforward.

Notice that a homomorphism from a partial ring into a partial ring preserves the partial orderings $\leq_r$, $\leq_l$, and $\leq$. These partial orderings are studied in great detail in Krause (1996).

3. PRE-STATES ON PARTIAL RINGS

Notice that $(\mathrm{R}, \mathrm{R}^2, +)$, where R denotes the real numbers, is a partial ring.

Let $(U, \perp, \oplus)$ be a partial ring. In the sequel we assume that $U \neq \emptyset$. A homomorphism $\mu: U \to \mathrm{R}$ is called a *measure on U*, i.e., μ is an element in the real vector space R^U and, for all elements $u, v \in U$ with $u \perp v$,

$$\mu(u \oplus v) = \mu(u) + \mu(v)$$

We denote by $M(U)$ the set of all measures on U. Let τ be the product topology on R^U, a locally convex Hausdorff topology.

Lemma 3.1. Let $(U, \perp, \oplus)$ be a partial ring. The set $M(U)$ is a τ-closed subspace of R^U.

Proof. Clearly, $M(U)$ is a subspace of R^U and the τ-limit of a converging net $(\mu_\alpha)_\alpha$ of $M(U)$ lies in $M(U)$.

A measure μ is said to be *bounded* if there exists a positive real number k such that, for all elements u in U, $|\mu(u)| \leq k$. The set of all bounded measures on U, denoted by $W(U)$, is a subspace of $M(U)$.

A measure μ is called *positive* if, for all elements u in U, $\mu(u) \geq 0$. Let $J^+(U)$ be the collection of all bounded, positive measures on U. Notice that $J^+(U)$ is a $\tau|_{W(U)}$-closed cone in $W(U)$. The elements of $J(U) := J^+(U) - J^+(U)$ are called *Jordan measures on U*. The elements of $\Sigma(U) := \{\mu \in J^+(U): \mu(u) \leq 1, \forall u \in U\}$ are called *pre-states* on U, and $\Sigma(U)$ is referred to as the *pre-state space of U*. Finally, the elements of $\Omega(U) := \{\mu \in \Sigma(U): \exists u \in U$ such that $\mu(u) = 1\}$ are called *states on U*, and $\Omega(U)$ is referred to as the *state space of U*.

Lemma 3.2. Let $(U, \perp, \oplus)$ be a partial ring. The pre-state space $\Sigma(U)$ of U is a τ-compact convex subset of R^U and $J^+(U) = \mathrm{R}^+\Sigma(U)$.

Proof. The subset $\Sigma(U)$ is clearly convex. Notice that $\Sigma(U)$ coincides with the set $[0, 1]^U \cap M(U)$. Since $[0, 1]^U$ is τ-compact (Tychonov cube), it follows by Lemma 3.1 that $\Sigma(U)$ is τ-compact. The remaining assertion follows easily.

Let $(U, \perp, \oplus)$ be a partial ring. Since the subsets $\Sigma(U)$ and $-\Sigma(U)$ of $J(U)$ are convex, it follows that the convex hull $\mathrm{con}(\Sigma(U) \cup -\Sigma(U))$ of

$\Sigma(U) \cup -\Sigma(U)$ coincides with the set $\{t\mu - (1 - t)\nu\colon \mu, \nu \in \Sigma(U);\ t \in [0, 1]\}$. Since $[0, 1]\Sigma(U) \subseteq \Sigma(U)$, it follows immediately that the convex set $\mathrm{con}(\Sigma(U) \cup -\Sigma(U))$ is circled and, by Lemma 3.2, we conclude that it also is absorbing in the vector space $J(U)$. As a consequence, the mapping $\| \cdot \|_U\colon J(U) \to \mathrm{R}^+$ defined, for elements μ in $J(U)$, by $\|\mu\|U \cdot := \inf\{t > 0\colon \mu \in t{\cdot}\mathrm{con}(\Sigma(U) \cup -\Sigma(U))\}$ [Minkowski functional over $\mathrm{con}(\Sigma(U) \cup -\Sigma(U))$] is a seminorm on $J(U)$.

Lemma 3.3. Let $(U, \perp, \oplus)$ be a partial ring and let $\Sigma(U)$ be the pre-state space of U. The convex circled set $\mathrm{con}(\Sigma(U) \cup -\Sigma(U))$ is a τ-compact subset of R^U.

Proof. We define a mapping $\varphi\colon \Sigma(U) \times \Sigma(U) \times [0; 1] \to \mathrm{R}^U$, for elements μ, ν in $\Sigma(U)$ and t in the real interval $[0; 1]$, by $\varphi(\mu, \nu, t) := t\mu - (1 - t)\nu$. By Lemma 3.2, $\Sigma(U)$ is τ-compact and, therefore, by Tychonov's Theorem, $\Sigma(U) \times \Sigma(U) \times [0; 1]$ is a compact subset of the space $\mathrm{R}^U \times \mathrm{R}^U \times \mathrm{R}$, equipped with the product topology determined by τ on R^U and the standard euclidean topology on R. Clearly, φ is a continuous map and, consequently, its range, which coincides with $\mathrm{con}(\Sigma(U) \cup -\Sigma(U))$, is a τ-compact subset of R^U.

Theorem 3.4. Let $(U, \perp, \oplus)$ be a partial ring. Let $J(U)$ ($\subseteq R^U$) be the real vector space of Jordan measures and let $J^+(U)$ be the cone of bounded positive measures on U. Then:

(i) The map $\| \cdot \|_U\colon J(U) \to \mathrm{R}^U$ defined above is a norm.

(ii)The topology determined by the norm $\| \cdot \|_U$ is finer than the restriction $\tau|_{J(U)}$ of the product topology τ on R^U to the subspace $J(U)$.

(iii) The normed linear space $(J(U), \| \cdot \|_U)$ is a Banach space.

(iv) The cone $J^+(U)$ is $\| \cdot \|_U$-closed.

(v) For every element μ in $J^+(U)$, $\|\mu\|_U = \sup_{u \in U}\mu(u)$. In particular, $\Sigma(U) = \{\mu \in J^+(U)\colon \|\mu\|_U \leq 1\}$ and every element in $\Omega(U)$ is of norm one.

(vi) For every element μ in $J(U)$ there exist elements λ, ν in $J^+(U)$ such that $\mu = \lambda - \nu$ and $\|\mu\|_U = \|\lambda\|_U + \|\nu\|_U$.

Proof. (i) Let μ be an element in $J(U)$. If $\|\mu\|_U$ is equal to zero, then so is $\|n\mu\|_U$, for all natural numbers n. Since $\mathrm{con}(\Sigma(U) \cup -\Sigma(U))$ is circled, we conclude that the element $n\mu$ lies in $1 \cdot \mathrm{con}(\Sigma(U) \cup -\Sigma(U))$. It follows that, for all natural numbers n, the set $n\mu(U)$ is contained in the real interval $[-1; +1]$. This shows that $\mu(U)$ coincides with $\{0\}$ or, equivalently, μ is equal to zero. Therefore the seminorm $\| \cdot \|_U$ is a norm.

(ii) Let $J(U)_1$ be the unit ball with respect to the norm $\| \cdot \|_U$ and let $J(U)_1^\circ$ be its interior. It follows that

$$J(U)_1^\circ \subseteq \mathrm{con}(\Sigma(U) \cup -\Sigma(U)) \subseteq J(U)_1 \tag{1}$$

Now let N be a zero-neighborhood in $J(U)$ with respect to the locally convex topology $\tau|_{J(U)}$. By Lemma 3.3, the subset $\mathrm{con}(\Sigma(U) \cup -\Sigma(U))$ is $\tau|_{J(U)}$-bounded. Therefore there exists a real number $t > 0$ such that $\mathrm{con}(\Sigma(U) \cup -\Sigma(U))$ is contained in tN. By (1), the $\|\cdot\|_{J(U)}$-open set $(1/t)J(U)_1^\circ$ is a subset of N. It follows that N is a zero-neighborhood in the norm-topology.

(iii) Let $(\mu_n)_{n\in\mathrm{N}}$ be a Cauchy sequence in the normed linear space $(J(U), \|\cdot\|_U)$. There exists a real number $s > 0$ such that, for all natural numbers n, μ_n lies in $sJ(U)_1$. By Lemma 3.3 (ii) and (1), $sJ(U)_1$ is a τ-compact subset of R^U. Therefore there exists a subnet $(\nu_\gamma)_{\gamma\in\Gamma}$ of $(\mu_n)_{n\in\mathrm{N}}$ which τ-converges to an element ν in $sJ(U)_1$. Specificially, there exists a map g from the upward-directed set Γ into N with the properties that (α) $\nu_\gamma = \mu_{g(\gamma)}$ and (β) for every element l in N, there exists an element $\gamma(l)$ in Γ such that, for all elements γ in Γ with $\gamma \geq \gamma(l)$, $g(\gamma) \geq l$ holds true.

Let $\varepsilon > 0$. Then there exists a natural number m such that, for all elements n in N with $n \geq m$, μ_n lies in the subset $\mu_m + (\varepsilon/2)J(U)_1$. Select an element $\gamma(m)$ in Γ satisfying condition (β) above. It follows that, for all elements γ in Γ with $\gamma \geq \gamma(m)$, $\nu_\gamma = \mu_{g(\gamma)}$ lies in $\mu_m + (\varepsilon/2)J(U)_1$. Since $J(U)_1$ is τ-closed and the net $(\nu_\gamma)_{\gamma\in\Gamma}$ τ-converges to ν, we conclude that ν is an element in $\mu_m + (\varepsilon/2)J(U)_1$. Hence, for all natural numbers n with $n \geq m$,

$$\|\nu - \mu_n\|_U \leq \|\nu - \mu_m\|_U + \|\mu_m - \mu_n\|_U = \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Therefore $(\mu_n)_n$ $\|\cdot\|_U$-converges to ν.

(iv) This follows by (ii) and the fact that $J(U)^+$ is $\tau|_{J(U)}$-closed.

(v) Let μ be an element in the cone $J^+(U)$ and let τ be equal to $\sup_{u\in U}\mu(u)$. Clearly, μ is an element in $r\Sigma(U)$, hence $\|\mu\|_U \leq r$. On the other hand, let $s > 0$ and suppose that μ lies in $s \cdot \mathrm{con}(\Sigma(U) \cup -\Sigma(U))$. Then there exist elements λ, ν in $\Sigma(U)$ and an element t in the interval [0; 1] such that μ/s is equal to $t\lambda - (1 - t)\nu$. Then, for all elements u in U, $\mu(u) = st\lambda(u) - s(1 - t)\nu(u) \leq s$, thus $\tau \leq s$. This shows that $\tau \leq \|\mu\|_U$.

(vi) Let μ be an element in $J(U)$ of norm one. By a remark above, μ lies in $\mathrm{con}(\Sigma(U) \cup -\Sigma(U))$. Therefore, there exist elements λ, ν in $\Sigma(U)$ and an element t in the interval [0; 1] such that μ is equal to $t\lambda - (1 - t)\nu$. Then

$$1 = \|\mu\|_U \leq \|t\lambda\|_U + \|(1 - t)\nu\|_U = t\|\lambda\|_U + (1 - t)\|\nu\|_U \leq 1$$

By Lemma 3.2, the elements $t\lambda$ and $(1 - t)\nu$ lie in $J^+(U)$.

Notice that, by Theorem 3.4(vi), the ordered normed vector space $(J(U), J^+(U), \|\cdot\|_U)$ associated with a partial ring $(U, \perp, \oplus)$ is 1-generated (Asimov and Ellis, 1980). Using the results of the next section, examples of partial rings can be constructed for which the associated ordered normed vector space of Jordan measures is not a base norm space. However, we have the following result:

Proposition 3.5. Let $(U, \perp, \oplus)$ be a partial ring. If U possesses a left or a right unit element, then the ordered normed vector space $(J(U), J^+(U), \|\cdot\|_U)$ is a complete base norm space and the state space $\Omega(L)$ of U is a base of the cone $J^+(U)$.

Proof. Suppose that U admits a right unit 1. Let u be an element in U. Then there exists an element ν in U such that $u \perp \nu$ and $u \oplus \nu$ is equal to 1. Therefore, for every element μ in $J^+(U)$,

$$\mu(u) \leq \mu(u) + \mu(v) = \mu(u \oplus v) = \mu(1) \tag{2}$$

If μ lies in $\Omega(U)$, then there exists an element w in U such that $\mu(w)$ is equal to one. By (2), it follows that $\mu(1)$ equals 1. On the other hand, if μ lies in $J^+(U)$ and $\mu(1)$ equals one, then, by (2), μ is contained in $\Sigma(U)$ and, therefore, in $\Omega(U)$. Consequently, $\Omega(U)$ coincides with the set $\{\mu \in J^+(L)$: $\mu(1) = 1\}$, which is clearly convex. Let μ be a nonzero element in $J^+(U)$. Then $\mu(1)$ is different from 0; else, by (2), $\mu(u)$ vanishes for all elements u in U. It follows that $\mu/\mu(1)$ lies in $\Omega(U)$. Therefore, $J^+(U)$ coincides with $\mathrm{R}^+\Omega(U)$ and $\Sigma(U)$ is equal to $[0; 1]\Omega(U)$.

4. THE PARTIAL RING OF TRIPOTENTS IN A JBW*-TRIPLE

Let A be a complex vector space. A *triple product* on A is a map $\{...\}$: $A \times A \times A \to A$ which is symmetric-bilinear in the outer variables and conjugate-linear in the middle variable. The following 'polarization formulas' are valid for elements a, b, c in A:

$$2\{a\ b\ c\} = \{(a + c)b(a + c)\} - \{a\ b\ a\} - \{c\ b\ c\} \tag{3}$$

$$4\{a\ b\ a\} = \sum_{k=0}^{3} (-1)^k\{b + i^k a\ b + i^k a\ b + i^k a\} \tag{4}$$

An element u in A is said to be a *tripotent* provided that u is equal to $\{u\ u\ u\}$. By $\mathcal{U}(A)$ we denote the collection of tripotents in A. Clearly, the zero vector 0 is a tripotent.

For a pair (a, b) of elements in A we define a linear map $D(a, b)$: $A \to A$, for elements c in A, by $D(a, b)(c) := \{a\ b\ c\}$. The map $D(a, b)$ is called the *multiplication operator associated with the pair* (a, b). For an element a in A the conjugate linear map $Q(a)$: $A \to A$ defined, for elements b in A, by $Q(a)(b) := \{a\ b\ a\}$ is called the *quadratic operator associated with a.*

The following theorem is due to W. Kaup.

Theorem 4.1. Let $(A, \| \cdot \|)$ be a complex Banach space. Then there is at most one triple product $\{\ldots\}: A \times A \times A \to A$ on A satisfying, for elements a, b, c, d, and e in A,

$$\{\{a\ d\ e\}b\ c\} - \{a\ d\{e\ b\ c\}\} = \{\{a\ b\ c\}d\ e\} - \{a\{b\ c\ d\}e\} \tag{5}$$

and satisfying the following conditions, for all elements a, b, c, in A:

(i) $\|\{a\ b\ c\}\| \leq \|a\|\cdot\|b\|\cdot\|c\|$ [in particular, the triple product and the mappings $D(a, b)$, $Q(a)$ are continuous].

(ii) $\|\{a\ a\ a\}\| = \|a\|^3$.

(iii) The linear operator $D(a, a)$ is hermitian [i.e., for every real number t, the norm of the bounded operator $\exp(itD(a, a))$ is equal to one].

(iv) The linear operator $D(a, a)$ has positive spectrum.

Proof. See Kaup (1983).

A complex Banach space which admits such a triple product is said to be a *JB*-triple.* The classes of C*-algebras, J*-algebras, and (nonassociative) JB*-algebras are examples of complex Banach spaces which are JB*-triples.

A JB*-triple A which is the Banach space dual of a (necessarily unique) Banach space A_* is called a *JBW*-triple.* Examples of complex Banach spaces which are JBW*-triples are given by W*-algebras, Hilbert spaces, spin triples, (nonassociative) JBW*-algebras, M_3^8, the hermitian 3×3 matrices with Cayley number entries with respect to the spectral norm, $B_{1,2}^8$, the 1×2 matrices with Cayley number entries with respect to the spectral norm, etc. Notice that in a W*-algebra B the triple product satisfying the conditions of Theorem 4.1 is given, for elements a, b, and c in B, by $\{a\ b\ c\} = (ab^* c + cb^* a)/2$. Therefore, an element u in B is a tripotent if and only if it is a partial isometry, i.e., uu^* and u^*u are self-adjoint idempotents in B.

In a JBW*-triple A the triple product is separately weak*-continuous. Consequently, the multiplication operators and the quadratic operators are weak*-continuous as well. The weak*-closure of the linear hull of the collection $\mathcal{U}(A)$ of tripotents in A coincides with A.

For each tripotent u in the JBW*-triple A, the weak*-continuous linear operators $P_j(u)$, $j = 0, 1, 2$, are defined, for each element a in A, by

$$P_2(u) = Q(u)^2;$$

$$P_1(u) = 2(D(u, u) - Q(u)^2),$$

$$P_0(u) = \mathrm{id}_A - 2D(u, u) + Q(u)^2$$

The linear operators $P_j(u)$, $j = 0, 1, 2$ are projections onto the eigenspaces $A_j(u)$ of $D(u, u)$ corresponding to eigenvalues $j/2$ and

$$A = A_0(u) \oplus A_1(u) \oplus A_2(u)$$

is the *Peirce decomposition* of A relative to u. For i, j, k equal to 0, 1, or 2, the *Peirce multiplication rules* hold:

$$\{A_j(u)A_k(u)A_l(u)\} \subseteq A_{j-k+l}(u)$$

when $j - k + l$ is equal to 0, 1, or 2, and

$$\{A_j(u)A_k(u)A_l(u)\} = \{0\} \tag{6}$$

otherwise. Moreover,

$$\{A\ A_2(u)\ A_0(u)\} = \{A\ A_0(u)\ A_2(u)\} = \{0\} \tag{7}$$

With respect to the separately weak*-continuous product $(a, b) \mapsto a \circ b = \{a\ u\ b\}$ and the norm-preserving involution $a \mapsto a^{\dagger} = \{u\ a\ u\}$, $A_2(u)$ is a JBW*-algebra with unit u. For details see Battaglia (1991), Barton *et al.*, (1987), Barton and Friedman (1987), Dineen (1986), Edwards *et al.* (1993, 1996), Edwards and Rüttimann (1992, 1996), Friedman and Russo (1985, 1986) Hanche-Olsen and Størmer (1984), Loos (1975), Neher (1987), and Wright (1977).

A pair u, v of elements in $\mathcal{U}(A)$ is said to be *orthogonal*, denoted by $u \perp v$, if v is contained in $A_0(u)$. It can be seen that $\perp$ is a symmetric binary relation on $\mathcal{U}(A)$. Moreover, $u \perp v$ if and only if $\{u\ u\ v\} = 0$ if and only if $D(u, v) = 0$.

Let A be a JBW*-triple. It follows immediately that, for a pair u, v in $\mathcal{U}(A)$ with $u \perp v$, $u + v$ lies again in $\mathcal{U}(A)$. We define a map $\oplus: \perp \to \mathcal{U}(A)$, for an orthogonal pair (u, v) of tripotents, by $u \oplus v = u + v$. The partial ring $(\mathcal{U}(A), \perp, \oplus)$ is referred to as the *partial ring of tripotents* of the JBW*-triple A.

Theorem 4.2. Let A be a JBW*-triple. Then the partial ring of tripotents $(\mathcal{U}(A), \perp, \oplus)$ of A is symmetric and satisfies both the cancellation and the associative law. Moreover, the zero vector 0 is an indecomposable neutral element.

Proof. It is easily seen that the partial ring $\mathcal{U}(A)$ is symmetric and satisfies the cancellation law. Let u, v, and w be tripotents. Suppose that $u \perp v$ and $(u \oplus v) \perp w$. It follows, by a remark above, that $\{v\ v\ w\} = \{u\ u \oplus v\ w\} = D(w, u \oplus v)(u) = 0$, and, therefore, $v \perp w$. Similarly, it follows that $u \perp w$. Consequently, $\{u\ u\ v \oplus w\} = \{u\ u\ v\} + \{u\ u\ w\} = 0$, hence, $u \perp (v \oplus w)$. Clearly, $(u \oplus v) \oplus w = u + v + w = u \oplus (v \oplus w)$. The zero vector 0 is a neutral element in $\mathcal{U}(A)$. In fact, 0 is indecomposable. To see this, suppose that (u, v) is an orthogonal pair in $\mathcal{U}(A)$ and that the sum $u \oplus v$ coincides with 0. Then $0 = \{u\ u\ 0\} = \{u\ u\ u\} + \{u\ u\ v\} = u$.

Proposition 4.3. Let A be a JBW*-triple and let $(\mathcal{U}(A), \perp, \oplus)$ be the partial ring of tripotents in A. Then $\mathcal{U}(A)$ possesses a right unit 1 if and only if A coincides with $\{0\}$.

Proof. Suppose that $\mathcal{U}(A)$ has a right unit 1. Let u be any tripotent in A. Then also $-u$ is a tripotent in A. Therefore there exist tripotents v, w such that $u \perp v$ and $-u \perp w$ and $u \oplus v = 1 = -u \oplus w$. Then

$$u = \{u\ u\ u \oplus v\} = \{u\ u\ 1\} = \{-u\ -u\ 1\} = \{-u\ -u\ -u + w\} = -u$$

Hence, u is equal to 0. Since A coincides with the weak*-closure of the linear hull of $\mathcal{U}(A)$, it follows that A coincides with $\{0\}$.

Theorem 4.4. Let A be a JBW*-triple and let $(\mathcal{U}(A), \perp, \oplus)$ be the partial ring of tripotents in A. Let u, v be elements in $\mathcal{U}(A)$. Then TFAE:

(i) $u \leq v$.
(ii) $\{u\ v\ u\} = \{v\ u\ v\} = u$.
(iii) $\{u\ v\ u\} = u$.

Proof. (i) $\Rightarrow$ (ii): If $u \leq v$, then there exists a tripotent w such that $u \oplus w$ is equal to v. Then

$$\{u\ v\ u\} = \{u\ u + v\ u\} = u + \{u\ w\ u\} = u$$

and

$$\{v\ u\ v\} = \{u + w\ u\ u + w\} = \{u\ u\ u\} + 2\{u\ u\ w\} + \{w\ u\ w\} = u$$

(ii) $\Rightarrow$ (iii): This is trivial.

(iii) $\Rightarrow$ (i): This rather deep result was obtained in Friedman and Russo (1985, Corollary 1.7).

For further properties of the partially ordered set $(\mathcal{U}(A), \leq)$ see Battaglia (1991), and Edwards and Rüttimann (1988, 1995). Also, the triple $(\mathcal{U}(A), \leq, \perp)$ can be viewed as a generalized orthomodular poset (Krause, 1996; Mayet-Ippolito, 1991).

We now turn our attention to the pre-states on the partial ring of tripotents of a JBW*-triple.

Let A be a JBW*-triple. Let x be an element in A_*. By Edwards and Rüttimann (1988, Lemma 3.6) and Friedman and Russo (1985, Proposition 2), there exists a smallest tripotent $e(x)$ in the partially ordered set $(\mathcal{U}(A), \leq)$ such that $e(x)(x)$ is equal to $\|x\|$. Moreover, $P_2(e(x))_*(x) = x$ and the restriction of x to the JBW*-algebra $A_2(e(x))$ is a faithful normal positive functional.

For every norm-one element x in A_*, we define a mapping $\mu_x\colon \mathcal{U}(A) \to \mathbb{C}$, for elements u in $\mathcal{U}(A)$, by

$$\mu_x(u) = \{u\ u\ e(x)\}(x)$$

Theorem 4.5. Let A be a JBW*-triple. Then, for every element x of norm one in the pre-dual space A_*, the functional μ_x is a state on the partial ring $\mathcal{U}(A)$.

Proof. Let $\mathcal{B}(\mathcal{A})$ be the unital Banach algebra of bounded linear operators on A. Let x be a norm-one element in A_*. We define a linear functional z_x on $\mathcal{B}(\mathcal{A})$, for elements T in $\mathcal{B}(\mathcal{A})$, by $z_x(T) := (Te(x))(x)$. Since

$$|z_x(T)| \leq \|Te(x)\| \cdot \|x\| \leq \|T\| \cdot \|e(x)\| \leq \|T\| \tag{8}$$

it follows that z_x is norm-continuous and of norm less than or equal to one.

Let u be a tripotent in A. Since the bounded linear operator $D(u, u)$ is hermitian and positive, i.e., its numerical range is positive, we conclude, by Bonsall and Duncan (1971, §5, Lemma 2) and by (8), that $0 \leq z_x(D(u, u)) = \mu_z(u) \leq 1$. Clearly,

$$\mu_z(e(x)) = \{e(x)\ e(x)\ e(x)\}(x) = e(x)(x) = 1 \tag{9}$$

Let $u,\ v$ be elements in $\mathcal{U}(A)$ with $u \perp v$. Then

$$\begin{aligned}\mu_x(u \oplus v) &= (D(u \oplus v, u \oplus v)e(x))(x)\\ &= (D(u, u)e(x))(x) + 2\{u\ v\ e(x)\}(x) + (D(v, v)e(x))(x)\\ &= \mu_x(u) \oplus \mu_x(v)\end{aligned}$$

Proposition 4.6. Let A be a JBW*-triple and let $(\mathcal{U}(A), \perp, \oplus)$ be the partial ring of tripotents. For every nonzero element u in $\mathcal{U}(A)$ there exists a state μ on $\mathcal{U}(A)$ such that

$$\mu(u) = 1$$

Proof. Let u be a nonzero tripotent. By Edwards and Rüttimann (1988, Lemma 3.2), there exists an element x in A_* of norm one such that $u(x)$ is equal to 1. Therefore, $e(x) \leq u$. Since μ_x is an isotone functional on the partially ordered set $(\mathcal{U}(A), \leq)$, it follows that $1 = \mu_x(e(x)) \leq \mu_x(u) \leq 1$.

Theorem 4.7. Let A be a JBW*-triple. Let x be an element in A_* of norm one. Let $(u_i)_{i\in I}$ be a family of tripotents in A and suppose that the supremum $\vee_{i\in I}u_i$ of this family exists in $(\mathcal{U}(A), \leq)$. If, for all elements i in I,

$$\mu_x(u_i) = 0$$

then

$$\mu_x\left(\bigvee_{i\in I} u_i\right) = 0$$

Proof. For any element i in I and j equal to 0, 1, or 2, denote by $u_{i,j}$ the element $P_j(e(x))u_i$. By the Peirce rules (6), (7), it follows that

$$\begin{aligned} 0 &= \{u_i u_i e(x)\}(x) \\ &= \{u_{i,2} u_{i,2} e(x)\}(x) + \{u_{i,1} u_{i,2} e(x)\}(P_2(e(x))_* x) \\ &\quad + \{u_{i,1} u_{i,1} e(x)\}(x) + \{u_{i,0} u_{i,1} e(x)\}(P_2(e(x))_* x) \\ &= (u_{i,2} \circ_{e(x)} u_{i,2}^{\dagger e(x)})(x) + \{u_{i,1} u_{i,1} e(x)\}(x) \end{aligned}$$

Clearly, the element $u_{i,2} \circ_{e(x)} u_{i,2}{}^{\dagger e(x)}$ lies in the positive cone $A_2(e(x))_+$ of the JBW*-algebra $A_2(e(x))$. By Friedman and Russo (1985, Lemma 1.5), the same holds true for the element $\{u_{i,1}\ u_{i,1}\ e(x)\}$. Since the restriction of x to $A_2(e(x))$ is a faithful normal state, we conclude that

$$u_{i,2} \circ_{e(x)} u_{i,2}^{\dagger e(x)} = \{u_{i,1} u_{i,1} e(x)\} = 0$$

Then $u_{i,2}$ is equal to zero and, again by Friedman and Russo (1985, Lemma 1.5), the same holds for $u_{i,1}$. Hence, for all elements i in I, the tripotent u_i lies in $A_0(e(x))$ or, equivalently, $u_i \perp e(x)$. By Battaglia (1991, Corollary 3.10), it follows that $\vee_{i \in I} u_i \perp e(x)$. Therefore, by (7),

$$\mu_x\left(\bigvee_{i \in I} u_i\right) = \left\{\bigvee_{i \in I} u_i \bigvee_{i \in I} u_i e(x)\right\}(x) = 0$$

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